ON THE INTERSECTION OF MAXIMAL m-CONVEX SUBSETS

BY

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ABSTRACT

A subset S of a real linear space E is said to be m-convex provided $m \geqq 2$, there exist more than m points in S, and for each m distinct points of S at least one of the $\binom{m}{2}$ segments between these m points is included in S. In E, let xy denote the segment between two points x and y. For any point x in $S \subset E$, let $S_x = \{y: xy \subset S\}$. The kernel of a set S is then defined as $\{x \in S: S_x = S\}$. It is shown that the kernel of a set S is always a subset of the intersection of all maximal m-convex subsets of S. A sufficient condition is given for the intersection of all the maximal m-convex subsets of a set S to be the kernel of S.

Introduction

The convexity of the kernel holds for subsets of any (not necessarily finite dimensional) linear space over any ordered field. The kernel of a closed set is closed in any linear topological space. The kernel of a set S has been characterized as the intersection of all the maximal convex subsets of S [4]. The result obtained by intersecting all of the maximal starshaped subsets of a compact, simply-connected set in the plane is a set which is either starshaped or empty [2]. Recently, it has been shown that the intersection of all the maximal L_n subsets of a set S in a linear topological space is again an L_n set [3]. The purpose of this paper is to establish a similar result for the intersection of maximal m-convex subsets of a set, and in the process generalize Toranzos' result.

1. Preliminaries

The results of this paper apply generally to subsets of a linear topological space E. The segment xy between two points x and y in E is the set of all points in E of the form $\alpha x + (1-\alpha)y$, where $0 \leqq \alpha \leqq 1$. A set S is said to be m-convex provided $m \geqq 2$, there exist more than m points in S, and for each m distinct points of S at least one of the $\binom{m}{2}$ segments between these m points is included in S. An exactly m-convex set is one which is m-convex but not $(m-1)$-convex.

A set S is said to be m-convex relative to a set T provided that $m \geqq 2$, there are more than m points in S, and for each m distinct points $x_1\ x_2, \cdots, x_m$ in S there is a segment x_ix_j determined by two of these points such that the open segment $(x_ix_j) = x_ix_j \backslash \{x_i, x_j\}$ is a subset of T.

If $S \subset T$, then S is said to be strongly convex relative to T if and only if for each two points x and y in S $(xy) \subset T$ implies $xy \subset S$. Thus, any set is strongly convex relative to itself while it need not be convex, that is, 2-convex. The convexity of a set implies both strong relative convexity and relative convexity with respect to any set containing it. However, a relatively convex subset S even of a convex set T need not be strongly convex relative to T. For instance, let T be a square with interior and S be the union of the two diagonals of T. Moreover, it is not necessarily true that a maximal k-convex subset of an m-convex set T be strongly convex relative to T. As an illustration, let W be a square with interior and sides pq, qr, rs, and sp. Let m and n be such that $m \neq q \neq n$, $pq \subset pm$, and $rq \subset rn$. The set T, consisting of W together with $qm \cup qn$, is 4-convex. However, the set S, consisting of $pm \cup rn$, is a maximal 3-convex subset of T which is not strongly convex relative to T.

If $S \subset E$ and $x \in S$, let $S_x = \{y \in S: xy \subset S\}$. S_x is called the x-star of S. The kernel of a set S, denoted by $\ker S$ is defined as $\{x \in S: S_x = S\}$. If $S \subset E$ and $x \in S$, let $S^x = \{y \in S: xy \not\subset S\}$. S^x is called the x-antistar of S. If S is closed and $x \in S$, then S^x is relatively open with respect to S, and if S is m-convex, then S^x is $(m-1)$-convex relative to S.

For convenience, we adopt the terminology that a subset $V = \{v_1, \cdots, v_t\}$ of a set S is visually independent relative to S if for all i and j such that $1 \leqq i < j \leqq t$, $v_iv_j \not\subset S$. The join of x and S is the set $xS = \{\alpha x + (1-\alpha)s: 0 \leqq \alpha \leqq 1, \text{and } s \in S\}$. This set is sometimes referred to as the cone over S with vertex x.

2. Maximal m-convex subsets

The first result together with Zorn's lemma will be used to establish the existence of certain maximal m-convex subsets of a given set.

THEOREM 1. *The union of a family of m-convex sets which is directed by* $\subset$ *(the union of any two members is contained in a third) is m-convex.*

PROOF. Let $F = \{C_a : a \in A\}$ be such a family and consider $B = \bigcup \{C_a : a \in A\}$. Select any m points $p_1, \cdots, p_m$ in B. Suppose p_i is in $C_{a(i)}$, for $1 \leqq i \leqq m$. By an inductive argument there is a set C_b such that $C_{a(i)} \subset C_b$, for $1 \leqq i \leqq m$. Hence, $p_1, \cdots, p_m$ are in C_b, and since C_b is m-convex the p_i determine at least one segment in C_b, which will also be a segment in B. Thus, B is m-convex.

LEMMA. 2 *If S is closed and M is a relatively m-convex subset of S, then the closure of M is m-convex relative to S.*

PROOF. Select any m points, $x_1 \cdots, x_m$ in $\mathrm{cl}\, M$, the closure of M, which are visually independent relative to S. Since S is closed, we may choose neighborhoods $U_j(x_i)$ and $U_i(x_j)$ of x_i and x_j respectively with the property that $uv \not\subset S$ for any $u \in U_j(x_i)$ and $v \in U_i(x_j)$. Let $U_i = \bigcap_{j \neq i} U_j(x)$. Now from the construction of the U_i, if y_i is a point in $M \cap U_i$, then $y_1, \cdots, y_m$ form a set of m points in M visually independent relative to S. Contradicting the m-convexity of M.

THEOREM 3. *Suppose S is a set which contains $m-1$ visually independent points, and let T be a subset of S, r-convex with respect to S, where $2 < r \leqq m$. Then there exists a set M which is maximal among all relatively m-convex subsets of S which include T. M is exactly m-convex with respect to S and if S is closed, M is closed.*

PROOF. If T is a relatively r-convex subset of S, it will be a relatively exactly s-convex subset of S relative to S for some $2 \leqq s \leqq r$. Let $x_1, \cdots, x_{m-1}$ be a set of $m-1$ visually independent points in S. Consider the sets $T_0 = T$, $T_1 = T \cup \{x_1\}$, $T_2 = T \cup \{x_1, x_2\}, \cdots, T_{m-1} = T \cup \{x_i : i = 1, \cdots, m-1\}$. At least one of these sets, say T_j must be exactly m-convex relative to S, since T_0 is relatively exactly s-convex, T_{m-1} is relatively exactly t-convex for some $m \leqq t \leqq s + m - 1$, and the addition of a point in S to each T_i does not increase the order of relatively exact m-convexity of T_i by more than one. By Zorn's lemma, there is a maximal subset M of S containing T_j which is m-convex relative to S. But since T_j is exactly m-convex relative to S, it contains $m-1$ points $y_1, \cdots, y_{m-1}$ which are visually independent relative to S. Since M contains $y_1, \cdots, y_{m-1}$, M itself is exactly m-convex relative to S. From Lemma 2 we have that if S is closed then M is closed and m-convex.

COROLLARY 4. *If T is a relatively 2-convex subset of an exactly m-convex* *set S, then there exists for each k, $2 \leqq k \leqq m$, a relatively k-convex maximal subset of S containing T.*

THEOREM 5. *If S is a set, then the intersection of any collection of (absolutely) m-convex subsets of S (m fixed, $m \geqq 2$) which are strongly convex relative to S is m-convex, provided the intersection contains at least m points.*

PROOF. Let $M = \cap \{S_i: i \in I\}$, where each S_i is an m-convex subset of S which is strongly convex relative to S. Choose any m distinct points in M, say $x_1, \cdots, x_m$. Note that $\{x_1 \cdots, x_m\} \subset S_i$ for all $i \in I$. If for some s and t, $1 \leqq s$, $t \leqq m$ and $u \in I$ $x_s x_t \subset S_u$, then $x_s x_t \subset S$ since $S_u \subset S$. Hence, $x_s x_t \subset S_i$ for all $i \in I$ by the strong relative convexity of S. Therefore, $x_s x_t \subset M$. Since S_u is m-convex, it must contain at least one segment $x_i x_j$ determined by two of the m points. Hence, M contains a segment determined by two of the given m points, and thus M is m-convex.

If S is an m-convex set $2 \leqq k \leqq m$, then its kernel is contained in any maximally relatively k-convex subset R of S. For if $x \in (\ker S) \backslash R$ then $\{x\} \cup R$ is clearly k-convex relative to S and contains R properly, a contradiction of the maximality of R.

THEOREM 6. *If R is any maximal (relatively) m-convex subset of S, then* $\ker S \subset R$.

PROOF. We prove this only for the case of m-convexity; the proof for the case of relative m-convexity if similar. Let R be a maximal m-convex subset of S and consider $x \in (\ker S) \backslash R$. R is a proper subset of xR and xR is m-convex. For, if we select any m distinct points $p_1, \cdots, p_m$ in xR, there exist points $x_1, \cdots, x_m$ in R such that $p_i \in xx_i$, $1 \leqq i \leqq m$. Furthermore, there is an s and t, $1 \leqq s, t \leqq m$ such that $x_s x_t$ is in R, since R is m-convex. Hence, $p_s p_t \subset \text{conv}\{x, x_s, x_t\} \subset xR$, contradicting the maximality of R. Thus, $\ker S \subset R$.

COROLLARY 7. *For any set S,* $\ker S$ *is a subset of the intersection of all the maximal (relatively) m-convex subsets of S.*

THEOREM 8. *Suppose S is a set with the property that S^x has at least $m-1$ visually independent points relative to S, for a fixed positive integer $m \geqq 2$ and every $x \in S \backslash \ker S$. Then* $\ker S$ *is the intersection of all the maximal exactly m-convex subsets of S.*

PROOF. Let $x \in S \backslash \ker S$. By hypothesis, there exist $m-1$ points $x_1, \cdots, x_{m-1}$ in S^x which are visually independent relative to S. The set $T = x_1(\ker S) \cup \cdots$

$\cup x_{m-1}(\ker S)$ is the union of $m-1$ convex subsets of S and hence is exactly m-convex relative to S. By Theorem 3, there exists a maximal m-convex subset M of S containing T. Now $x \notin M$, since $x, x_1, \cdots, x_{m-1}$ are visually independent relative to S, hence, visually independent relative to M. Therefore, x cannot be an element of the intersection of all maximal exactly m-convex subsets of S. Thus, the intersection of the maximal exactly m-convex subsets of S is a subset of $\ker S$. Using Coroll..ry 7, the theorem is est..blished.

COROLLARY 9. *Suppose S is an m-convex set with the property that, for a fixed positive integer k, $2 \leqq k \leqq m-1$, and for every $x \in S \setminus \ker S$, S^x is exactly k-convex relative to S. Then $\ker S$ is the intersection of all the maximal k-convex subsets of S.*

The four segments pq, qr, rs and st in the form of the letter W provide us with an example of a 5-convex set S, with the property that $\{q, r, s\}$ is the intersection of all the maximal 4-convex subsets of S and $\ker S = \varnothing$. Thus, the plausible conjecture

$$\ker S = \bigcap_{M \subset S} M \tag{1}$$

where the intersection is taken over all the maximal (relatively) m-convex subsets of S, is false. A more interesting counterexample to (1) is illustrated in Fig. 1.

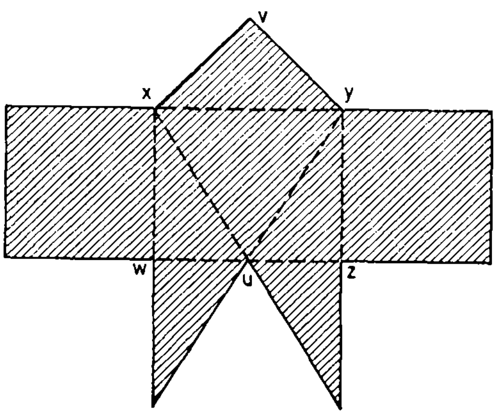

Fig. 1

The set S is compact, connected and 4-convex. However the kernel of S is not the intersection of the maximal 3-convex subsets of S. We have $\ker S = \text{conv}\{x, y, u\}$, and the intersection of the maximal 3-convex subsets of S is $\text{conv}\{x, y, z, w\}$. Note that S^v is convex relative to S.

On the positive side, Fig. 2 illustrates an example of a set S in the plane which satisfies the property required in Theorem 8 for each $m \geqq 2$; using complex, notation, S consists of a square B centered at the origin, and the union of the cones of the points $z(n,2j)$ and $z(n,2j+1)$ over B, $n = 1,2 \cdots$, and $j = 0,1,2,$ and 3, where $z(n,2j) = \exp(\pi j/2 - a + a/n - a/n^2)i$ and $z(n,2j+1) = \exp(\pi j/2 + a - a/n + a/n^2)i$, with a chosen so that $z(1,j)$, $j = 0,1,\cdots,7$, are the points of intersection of the sides of B and the unit circle $|z| = 1$. Here, $\ker S = B$ and, according to Theorem 8, B is obtained by intersecting all maximal, relatively exactly m-convex subsets of S, for each value of m.

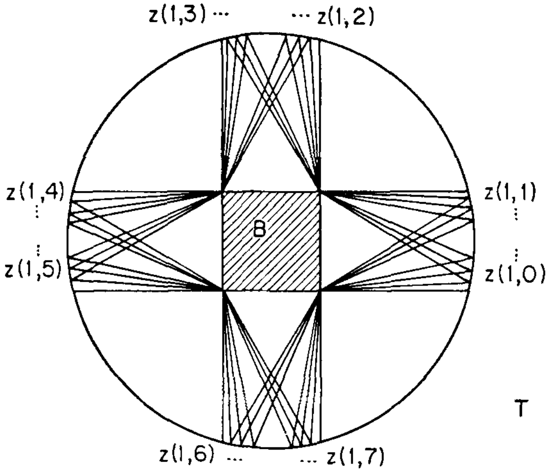

Fig. 2

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